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Return to Thermal Equilibrium by the Solution of a Quantum Langevin Equation

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A quantum-mechanical treatment of the evolution of an anharmonic oscillator coupled to a heat bath is given. It is shown that for a certain class of anharmonic potentials the heat bath drives the oscillator to an equilibrium state, close to the quantum Gibbs state associated to the potential. Thus a partial proof is provided for a conjecture of R. Benguria and M. Kac.

KEY WORDS: Langevin equation; return to equilibrium; mixing dynamical systems; von Neumann algebras.

1. INTRODUCTION

In 1908, Langevin proposed his equation for the description of Brownian motion. For the case of a particle on a line in an external potential v this equation is the following:

$$\frac{d^2}{dt^2}X_t + \eta\frac{d}{dt}X_t + v'(X_t) = (2\eta)^{1/2}E_t^\beta \tag{1.1}$$

Here, η is a friction coefficient, and E^β denotes what is now known as "white noise" of temperature $1/\beta$, the Gaussian generalized stochastic process with covariance given by

$$\langle E_t^\beta E_s^\beta\rangle = \beta^{-1}\delta(t-s)$$

In 1930 Uhlenbeck and Ornstein constructed the solution of the Langevin equation (1.1). Viewed as a stochastic process with values in the

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phase space of the particle, this solution is a Markov process. As a result of this property, time evolution acts as a semigroup of transformations on the space of all probability densities on the phase space. This semigroup is generated by a diffusion equation, the Fokker–Planck equation. If we suppose that the potential v is of a cuplike form, there is precisely one probability density left fixed under the time evolution, and all other densities flow towards it as time goes on. This one stationary probability density turns out to be the Gibbs probability distribution associated with the potential v. (Cf., for instance, Ref. 3.) In this sense the solution (1.1) returns to thermal equilibrium.

The probabilistic theory of Brownian motion being established, from the physical point of view two fundamental questions remained to be answered. On the one hand it was not clear whether (1.1) could be derived from microscopic considerations based on classical mechanics alone, and on the other hand some authors wondered what might be a suitable corresponding theory in quantum mechanics. Answers to both questions were provided by G. Ford, M. Kac, and P. Mazur, using a harmonic oscillator model. In 1965[2] they showed that, in a chain of coupled harmonic oscillators, one of the oscillators can be made to satisfy (1.1) [with $v(x) = \frac{1}{2}x^2$] to arbitrary accuracy by an appropriate choice of the coupling strengths. A quantum theory of friction and noise was now obtained by quantization of the oscillators in the chain. The quantum Langevin equation satisfied by an element of this chain, is formally identical with (1.1), but now X_t is a self-adjoint operator on the Hilbert space of the chain of oscillators, and E^β is an operator-valued distribution, satisfying the commutation relation

$$[E_t^\beta, E_s^\beta] = i\delta'(t-s)\mathbb{1} \tag{1.2}$$

It is a consequence of this relation, and of the assumption of thermal equilibrium for the entire chain at inverse temperature β, that the covariance of E^β is now given by

$$\langle E_t^\beta E_s^\beta \rangle = \int_{-\infty}^{\infty} \frac{k}{1-e^{-\beta k}} e^{-ik(t-s)} \frac{dk}{2\pi} \tag{1.3}$$

For the special case $v(x) = \frac{1}{2}x^2$, treated in Ref. 2, the quantum Langevin equation is linear and readily solved. Its stationary solution is the family $\{Q_t^{\beta,\eta}\}_{t\in\mathbb{R}}$ given by the formal expression

$$Q_t^{\beta,\eta} = \int_{-\infty}^{t} q_\eta(s-t) E_s^\beta \, ds \tag{1.4}$$

where $q_\eta : \mathbb{R} \to \mathbb{R}$ is zero on $(0, \infty)$, and on $(-\infty, 0]$ it is the solution of the differential equation $q'' - \eta q' + q = 0$ with boundary conditions $q(0) = 0$ and $q'(0) = -(2\eta)^{1/2}$. This solution $\{Q_t^{\beta,\eta}\}$ has the property of return to thermal equilibrium in the following sense. Suppose one perturbs the state of the entire system at time zero by changing the probability distribution of the oscillator and, locally in time, that of the incoming noise, thus bringing the system in the vector state ψ, say. We then have, irrespective of ψ,

$$\lim_{t\to\pm\infty} \langle \psi, \exp(i\lambda Q_t^{\beta,\eta})\psi\rangle$$
$$= \exp\left[-\tfrac{1}{2}\lambda^2 \int_{-\infty}^{\infty} \frac{k}{1-e^{-\beta k}} \frac{2\eta}{(k^2-1)^2 + \eta^2 k^2} \frac{dk}{2\pi}\right]$$

and in the "low-friction limit" $\eta \downarrow 0$ this tends to

$$\exp\left[-\tfrac{1}{4}\lambda^2(\coth\tfrac{1}{2}\beta)\right]$$

the expectation value of $\exp(i\lambda\,\cdot)$ in the Gibbs state of a quantum-mechanical harmonic oscillator at inverse temperature β. All the above results can be found in Ref. 2.

It is important to know whether this property of return to equilibrium is just a lucky consequence of the linearity of the equation, or whether it persists under perturbations. This question must necessarily be approached along different lines as were sketched above for the classical case, because the quantum stochastic process at hand is not a Markov process. In fact, quantum Markov processes cannot occur in thermal equilibrium.[(4)] (Markovian limits like the weak-coupling limit in rescaled time destroy the thermal equilibrium state.)

A few years ago, R. Benguria and M. Kac conjectured that for a class of potentials v the distribution of the stationary solution $\{X_t\}$ of the quantum Langevin equation approaches in the low friction limit $\eta\downarrow 0$ the quantum-mechanical Gibbs distribution of the oscillator subject to the potential v. They provided evidence to support this conjecture by a perturbation calculation carried through to third order.[(1)]

In this paper we formulate this and related questions in the setting of W^*-dynamical systems, and show that it is possible to give a precise answer to some of them.

We shall be concerned with the following questions: If in (1.1) we put

$$v(x) = \tfrac{1}{2}x^2 + w(x) \tag{1.5}$$

does it have a stationary solution $\{X_t\}_{t\in\mathbb{R}}$? If so, does the following limit exist for all ψ:

$$\lim_{t\to\infty} \langle \psi, \exp(i\lambda X_t)\psi\rangle =: \hat{\mu}_{\beta,\eta,w}(\lambda) \tag{1.6}$$

And do we have

$$\lim_{\eta \downarrow 0} \hat{\mu}_{\beta,\eta,w}(\lambda) = \hat{\mu}_{\beta,0,w}(\lambda) \tag{1.7}$$

where $\mu_{\beta,0,w}$ is the quantum-mechanical Gibbs measure, given in terms of the Hamiltonian $h = -\frac{1}{2}\partial^2/\partial x^2 + \frac{1}{2}x^2$ of the harmonic oscillator as follows:

$$\hat{\mu}_{\beta,0,w}(\lambda) = \frac{\operatorname{tr}\{\exp[-\beta(h+w)]\exp(i\lambda\cdot)\}}{\operatorname{tr}\{\exp[-\beta(h+w)]\}} \tag{1.8}$$

Our, partial, answers are based on the perturbation theory of W^*-dynamical systems. The Dyson series occurring in this perturbation theory turns out to be L^1-convergent for a limited class of perturbations w described by a certain inequality. If w is in this class, the quantum Langevin equation with v given by (1.5) indeed has a solution $\{X_t\}$, whose distribution returns to some equilibrium measure $\mu_{\beta,\eta,w}$ in the sense of (1.6). However, the above-mentioned class of perturbations shrinks to the class of constant functions as η decreases to zero, and we have a proof of (1.7) only in the trivial case that w is a constant, corresponding to the linear quantum Langevin equation. For more general w's we must content ourselves with an estimate of the difference between $\mu_{\beta,\eta,w}$ and $\mu_{\beta,0,w}$, which shows that at least these two measures can be brought arbitrarily close together by choosing η small, albeit for a narrowing class of w's.

In fact, the L^1-convergence of the Dyson series is more than one can hope for in a general W^*-dynamical system, whereas we feel that the stability of the property of return to equilibrium should be much more common. We expect, therefore, that our result is far from optimal.

This paper is organized as follows. In Section 2 a construction is given of the quantum noise, and the linear quantum Langevin equation is seen to have a stationary solution which can be viewed as a single operator, swept along by the flow of a strongly mixing W^*-dynamical system. In Section 3 methods are described for perturbing W^*-dynamical systems. It is shown in Section 4 that the L^1-convergence of the Dyson series implies that the unperturbed equilibrium state returns to the perturbed one under the perturbed flow. Computationally, this result comes closest to that of Ref. 1. In Section 5, another consequence of the L^1-convergence of the Dyson series is pointed out: the perturbed system is actually isomorphic to the unperturbed one, thus inheriting the strong mixing property. In Section 6 it is checked that indeed the Dyson series associated with our perturbation is L^1-convergent, provided that the anharmonic term in the potential satisfies a certain inequality. It is shown in Section 7 that this way of perturbing the dynamics indeed leads to a solution of the anharmonic Langevin equation.

Finally the behavior of the perturbed equilibrium state for small values of the coupling constant η is considered in Section 8.

Let us remark here that the present program does not apply without change to quantum noise of zero temperature. The associated dynamical system is not strongly mixing, and there does not necessarily exist a normal state, invariant for the perturbed dynamics (cf. Ref. 5).

2. CONSTRUCTION OF THE DYNAMICAL SYSTEM

Since all E^β satisfying (1.2) and (1.3) are unitarily equivalent, we shall not bother about the details of the model that produced these relations, but instead construct a simple version of E^β.

Let $\mathscr{S}$ be Schwartz's class of rapidly decreasing, infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$, and let σ be the symplectic form on $\mathscr{S}$, given by

$$\sigma(f, g) = \int_{-\infty}^{\infty} fg' \, dt$$

On the CCR algebra $\mathscr{W}$ over the symplectic space $\{\mathscr{S}, \sigma\}$, the state ω_β, given by

$$\omega_\beta(W(f)) = \exp\left(-\tfrac{1}{2}\|f\|_\beta^2\right)$$

with

$$\|f\|_\beta^2 = \int_{-\infty}^{\infty} \frac{k}{1 - e^{-\beta k}} |\hat{f}(k)|^2 \frac{dk}{2\pi} \tag{2.1}$$

satisfies the KMS condition with respect to the *-automorphism group α of $\mathscr{W}$, corresponding to the translations on $\mathscr{S}$:

$$\alpha_t(W(f)) = W(T_t f), \qquad (T_t f)(s) = f(s - t)$$

We thus use the idea of a translation representation, introduced into this context by Lewis and Thomas.[6] Cf. also Ref. 7.

Now, let $\{\mathscr{H}_\beta, \pi_\beta, \xi_\beta\}$ be the GNS-triple associated with $\mathscr{W}$ and ω_β, and let $\mathscr{M}_\beta$ be the closure of $\pi_\beta(\mathscr{W})$ in the strong operator topology. Let us identify $\mathscr{W}$ with $\pi_\beta(\mathscr{W}) \subset \mathscr{M}_\beta$, and extend ω_β and α to all of $\mathscr{M}_\beta$ in the natural way. Then $\{\mathscr{M}_\beta, \omega_\beta, \alpha\}$ is a strongly mixing W^*-dynamical system in thermal equilibrium at inverse temperature β in the sense of the following definition:

Definition. By a *W^*-dynamical system* we mean a triple $\{\mathscr{M}, \omega, \alpha\}$, where $\mathscr{M}$ is a von Neumann algebra, ω a normal state on $\mathscr{M}$, and $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ a weak *-continuous group of ω-preserving *-automorphisms of $\mathscr{M}$. If ω is faithful and $\{\alpha_{-\beta t}\}_{t \in \mathbb{R}}$ is its modular group, we say that $\{\mathscr{M}, \omega, \alpha\}$ is *in thermal equilibrium* (*at inverse temperature* β). If for all normal states ϕ on $\mathscr{M}$ and all $M \in \mathscr{M}$

$$\lim_{t \to \pm\infty} \phi(\alpha_t(M)) = \omega(M) \tag{2.2}$$

we call $\{\mathscr{M}, \omega, \alpha\}$ *strongly mixing*.

Next, define the self-adjoint operators $E^\beta(f)$, $(f \in \mathscr{S})$ on $\mathscr{H}_\beta$ by

$$\exp\left[i\lambda E^\beta(f)\right] = W(\lambda f) \qquad (\lambda \in \mathbb{R})$$

The map E^β from $\mathscr{S}$ to the operators on $\mathscr{H}_\beta$ can be continuously extended to the closure $\overline{\mathscr{S}}^\beta$ of $\mathscr{S}$ in the norm $\|\cdot\|_\beta$, given in (2.1). The function q_η introduced in (1.4) is in $\overline{\mathscr{S}}^\beta$, and therefore we may define the family $\{Q_t^{\beta,\eta}\}_{t\in\mathbb{R}}$ by

$$Q_t^{\beta,\eta} = E^\beta(T_t q_\eta)$$

It is not hard to show that, for all ψ in the linear span of $\{W(g)\xi_\beta \mid g \in \overline{\mathscr{S}}^\beta\}$ and all $f \in \mathscr{S}$ we have

$$\int_{-\infty}^{\infty} (f'' - \eta f' + f)(t) Q_t^{\beta,\eta} \psi \, dt = (2\eta)^{1/2} E^\beta(f)\psi \tag{2.3}$$

This is a distribution form of the Langevin equation (1.1) with potential $v(x) = \frac{1}{2}x^2$.

We note in passing that $p_\eta := d/dt(T_t q_\eta)|_{t=0}$ is not in $\overline{\mathscr{S}}^\beta$ for any $\beta > 0$, so that a momentum operator $P_t^{\beta,\eta}$ cannot be defined by putting

$$P_t^{\beta,\eta} := E^\beta(T_t p_\eta) \tag{2.4}$$

This indicates that the quantum-mechanical Uhlenbeck–Ornstein process, unlike its classical counterpart, moves about too wildly, due to vacuum fluctuations in the noise, to possess a momentum observable.

Now consider the Langevin equation with potential $v(x) = \frac{1}{2}x^2 + w(x)$ with w real and bounded. The key idea in solving this equation is to add a perturbation operator $V = w(Q_0^{\beta,\eta})$ to the Hamiltonian of the dynamical system. It turns out that, in the dynamics thus obtained, $Q_0^{\beta,\eta}$ satisfies the quantum Langevin equation with potential v and a transformed noise. The problem is to prove that the perturbed dynamical system is also mixing. In the next three sections, perturbations of W^*-dynamical systems will be considered in general.

3. PERTURBATIONS OF W^*-DYNAMICAL SYSTEMS

Let $\{\mathscr{M}, \omega, \alpha\}$ be a W^*-dynamical system in thermal equilibrium. Given $V = V^* \in \mathscr{M}$, there are two equivalent, but apparently different ways of perturbing it to obtain another W^*-dynamical system in thermal equilibrium, $\{\mathscr{M},\omega^V,\alpha^V\}$ say. One is called the time-dependent perturbation theory and perturbs α. It was first formulated, for general C^*-algebras, by Robinson.[8] The other is called the time-independent perturbation theory and perturbs ω. It was developed by Araki.[9] If $\mathscr{M}$ is a factor, the time evolutions and their equilibrium states are in one-to-one correspondence, and therefore the two approaches must be equivalent.

The time-dependent perturbation theory defines α^V by the Dyson series

$$\alpha_{-t}\circ\alpha_t^V(M) = \sum_{n=0}^{\infty} i^n \int_{0\geqslant t_1\geqslant\cdots\geqslant t_n\geqslant -t} dt_1\cdots dt_n \left[\alpha_{t_n}(V),\left[\cdots\left[\alpha_{t_1}(V),M\right]\cdots\right]\right] \tag{3.1}$$

The groups α and α^V satisfy the integral equations

$$\alpha_{-t}\circ\alpha_t^V(M) = M + i\int_0^t \alpha_{-s}\big(\left[V,\alpha_s^V(M)\right]\big)\,ds \tag{3.2}$$

and

$$\alpha_{-t}^V\circ\alpha_t(M) = M - i\int_0^t \alpha_{-s}^V\big(\left[V,\alpha_s(M)\right]\big)\,ds \tag{3.3}$$

(In fact, (3.1) was obtained by iteration of (3.2).)

The time-independent perturbation theory constructs an α^V-KMS-state ω^V in the following way: For each $\mathbf{M}=\{M_0,M_1,\ldots,M_n\}\in\mathscr{M}^{n+1}$ there is a unique bounded and continuous function $G_{\mathbf{M}}$ on the region

$$\Lambda_n^\beta := \left\{\{z_1,\ldots,z_n\}\in\mathbb{C}^n \,|\, 0\leqslant \operatorname{Im} z_1\leqslant\cdots\leqslant\operatorname{Im} z_n\leqslant\beta\right\} \tag{3.4}$$

which is analytic on the interior of Λ_n^β and such that for all $t_1,\ldots,t_n\in\mathbb{R}$,

$$G_{\mathbf{M}}(t_1,\ldots,t_n) = \omega\big(M_0\alpha_{t_1}(M_1)\cdots\alpha_{t_n}(M_n)\big)$$

The state ω^V is then given by

$$\omega^V(M) = \rho^V(M)/\rho^V(\mathbb{1})$$

where

$$\rho^V(M) = \sum_{n=0}^{\infty}(-1)^n\int_{0\leqslant s_1\leqslant\cdots\leqslant s_n\leqslant\beta} ds_1\cdots ds_n\, G_{M,V,\ldots,V}(is_1,\ldots,is_n) \tag{3.5}$$

We shall not need this expression for ω^V, however, until in Section 8 we consider the limit $\eta\downarrow 0$. The existence of ω^V suffices for the results in the next two sections.

4. RETURN TO THE PERTURBED EQUILIBRIUM STATE

The Dyson series on the right-hand side of (3.1) always converges in norm for all finite values of t, but rarely does it converge for $t = \infty$. In the special case described in Section 2, this happens to be the case, however.

If V and A are operators in $\mathscr{M}$, and $t_1, \ldots, t_n$ are real numbers, let $V(A; t_1, \ldots, t_n)$ denote the operator

$$\left[\alpha_{t_n}(V), \left[\cdots\left[\alpha_{t_1}(V), A\right]\cdots\right]\right] \in \mathscr{M}$$

Definition. We shall say that the Dyson series on the right-hand side of (3.1) is L^1-*convergent* if

$$\sum_{n=0}^{\infty} \int_{0 \geq t_1 \geq \cdots \geq t_n > -\infty} dt_1 \cdots dt_n \, \| V(A; t_1, \ldots, t_n) \| < \infty \tag{4.1}$$

We postpone the proof of the L^1-convergence of the Dyson series to Section 6. Here we state some of its consequences.

Theorem 4.1. Let the strongly mixing W^*-dynamical system in thermal equilibrium $\{\mathscr{M}, \omega, \alpha\}$ be perturbed to $\{\mathscr{M}, \omega^V, \alpha^V\}$. Suppose that the Dyson series involved is L^1-convergent for all elements of some subset $\mathscr{A}$ of $\mathscr{M}$.

Then for all $A \in \mathscr{A}$ and all $M \in \mathscr{M}$:

$$\lim_{t \to \infty} (\omega^V - \omega)\big(\alpha_t(M)\alpha_t^V(A)\big) = 0 \tag{4.2}$$

Proof. Using (3.1) we may write

$$\begin{aligned}
&(\omega^V - \omega)\big(\alpha_t(M)\alpha_t^V(A)\big) \\
&\quad = \omega^V\big(\alpha_t\big(M\alpha_{-t} \circ \alpha_t^V(A)\big)\big) - \omega\big(M\alpha_{-t} \circ \alpha_t^V(A)\big) \\
&\quad = \sum_{n=0}^{\infty} i^n \int_{0 \geq t_1 \geq \cdots \geq t_n > -\infty} dt_1 \cdots dt_n \\
&\qquad \times \theta(t + t_n)\big(\omega^V\big(\alpha_t\big(MV(A; t_1, \ldots, t_n)\big)\big) \\
&\qquad\qquad - \omega\big(MV(A; t_1, \ldots, t_n)\big)\big)
\end{aligned} \tag{4.3}$$

Here, θ is Heaviside's function.

By the mixing property of $\{\mathscr{M}, \omega, \alpha\}$, the integrand in (4.3) tends to zero for all $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \in \mathbb{R}$. Its absolute value is bounded by the function $2\|M\| \cdot \|V(A; t_1, \ldots, t_n)\|$, which is in L^1 by assumption. The statement (4.2) now follows by the dominated convergence theorem. ■

Corollary 4.2. Under the same assumptions, for all $A \in \mathscr{A}$,

$$\lim_{t\to\infty} \omega \circ \alpha_t^V(A) = \omega^V(A) \tag{4.4}$$

Proof. Put $M = \mathbb{1}$ in (4.2). ■

The content of Corollary 4.2, combined with a consideration of the weak-coupling limit $\eta\downarrow 0$ for both sides of (4.4), was the kind of return to equilibrium, aimed at by Benguria and Kac.[1] For the special choices $\omega = \omega_\beta$, $V = \exp(aQ_0^{\beta,\eta})$, and $A = \exp(bQ_0^{\beta,\eta})$, the Dyson series for $\lim_{t\to\infty}\omega(\alpha_t^{\epsilon V}(A))$ and the power series in ϵ, obtainable for instance from (3.5), for $\omega^{\epsilon V}(A)$ were compared in the limit $\eta\downarrow 0$. A few terms were found to be equal.

As a matter of fact, no perturbation of the dynamics was considered in Ref. 1, neither was the Dyson series explicitly mentioned. Instead, the anharmonic Langevin equation was rewritten as an integral equation for the position operator, $X^{\epsilon V}$ say, in terms of $Q_0^{\beta,\eta}$. A recursive procedure was devised for the computation of the coefficients of the power series in ϵ of $\omega(\exp(bX^{\epsilon V}))$. One may check that this procedure yields precisely the Dyson series for $\lim_{t\to\infty}\omega \circ \alpha_t^{\epsilon V}(A)$.

In Ref. 1 the authors said they could not "escape the feeling that (their) calculations merely constitute an elaborate verification of the inner consistency of quantum mechanics." It turns out that this feeling was not deceptive. Indeed, Theorem 2 of Ref. 5 asserts that, if one does not bother about convergence of limits, the termwise identity of the two power series compared in Ref. 1 is a consequence of the KMS condition, together with the strong mixing property of $\{\mathscr{M}, \omega, \alpha\}$. These two requirements do not imply, however, that $\{\mathscr{M}, \omega^V, \alpha^V\}$ is mixing or indeed that (4.4) is valid, as is shown by a counterexample, given in Ref. 10.

Only if each of the coefficients and the entire series converge can it be concluded that their sums are equal.

5. ISOMORPHISM OF THE DYNAMICAL SYSTEMS

A much stronger statement can be proved from the L^1-convergence of the Dyson series: the W^*-dynamical systems $\{\mathscr{M}, \omega, \alpha\}$ and $\{\mathscr{M}, \omega^V, \alpha^V\}$ are actually isomorphic. As a consequence, $\{\mathscr{M}, \omega^V, \alpha^V\}$ is also mixing.

The proof which will be given here is based on the idea of Robinson[8] to consider the limit

$$\lim_{t\to\infty} \alpha_{-t}^V \circ \alpha_t(A) =: \gamma_0^V(A) \tag{5.1}$$

for all A in a properly chosen *-subalgebra $\mathscr{A}$ of $\mathscr{M}$. Under fairly mild conditions—in fact only the $n = 1$ term in (4.1) is needed—this limit exists and defines a *-morphism $\gamma_0^V : \mathscr{A} \to \mathscr{M}$ with the intertwining property

$$\gamma_0^V \circ \alpha_t = \alpha_t^V \circ \gamma_0^V \tag{5.2}$$

Moreover, by the mixing property, γ_0^V transforms ω^V to ω:

$$\omega^V \circ \gamma_0^V(A) = \lim_{t\to\infty} \omega^V \circ \alpha_t(A) = \omega(A) \tag{5.3}$$

What we have to do is to extend γ_0^V to all of $\mathscr{M}$, and to prove that it becomes surjective. In order to do the former, we have to go down to the Hilbert space level. For the latter we need the "inverse Møller" limit

$$\lim_{t\to\infty} \alpha_{-t} \circ \alpha_t^V(A) =: \tilde{\gamma}_0^V(A) \tag{5.4}$$

Any W^*-dynamical system $\{\mathscr{M}, \omega, \alpha\}$ brings along with it—or indeed has been constructed from—what we shall call a *Hilbert space dynamical system* $\{\mathscr{H}, \mathscr{M}, \xi, U\}$. Here, $\mathscr{H}$ is the GNS space associated with $\mathscr{M}$ and ω, with cyclic vector ξ. $\mathscr{M}$ is the concrete von Neumann algebra of operators on $\mathscr{H}$, and $U = \{U_t\}_{t\in\mathbb{R}}$ is the group of unitaries on $\mathscr{H}$, defined by

$$U_t M\xi = \alpha_t(M)\xi \qquad (M \in \mathscr{M})$$

Theorem 5.1. Let the strongly mixing W^*-dynamical system $\{\mathscr{M}, \omega, \alpha\}$ be perturbed to $\{\mathscr{M}, \omega^V, \alpha^V\}$ and let $\{\mathscr{H}, \mathscr{M}, \xi, U\}$ and $\{\mathscr{H}, \mathscr{M}, \xi^V, U^V\}$ be the associated Hilbert space dynamical systems. Suppose that for all A in some weak *-dense, α-invariant, unital sub-*-algebra $\mathscr{A}$ of $\mathscr{M}$ the following holds:

$$\int_{-\infty}^{0} \|[\alpha_t(V), A]\|\, dt < \infty \tag{5.5}$$

Then there is an isometry $\Omega^V : \mathscr{H} \to \mathscr{H}$ with the properties

$$\Omega^V \circ U_t = U_t^V \circ \Omega^V \tag{5.6}$$

and

$$\Omega^V \xi = \xi^V \tag{5.7}$$

Proof. From (3.3) it follows that for all $0 \leqslant s \leqslant t$

$$\|\alpha_{-t}^V \circ \alpha_t(A) - \alpha_{-s}^V \circ \alpha_s(A)\| \leqslant \int_s^t \|[\alpha_{-u}(V), A]\|\, du$$

and it follows from (5.5) that $t \mapsto \alpha_{-t}^{V} \circ \alpha_t(A)$ is Cauchy in the operator norm for $t \to \infty$. Therefore γ_0^V exists as a *-morphism $\mathscr{A} \to \mathscr{M}$. Now define

$$\Omega_0^V : \mathscr{A}\xi \to \mathscr{M}\xi^V : A\xi \mapsto \gamma_0^V(A)\xi^V$$

By (5.3) we have for all $A \in \mathscr{A}$,

$$\begin{aligned}\|\Omega_0^V A\xi\|^2 &= \|\gamma_0^V(A)\xi^V\|^2 = \langle \xi^V, \gamma_0^V(A)^*\gamma_0^V(A)\xi^V\rangle = \omega^V \circ \gamma_0^V(A^*A)\\ &= \omega(A^*A) = \langle \xi, A^*A\xi\rangle = \|A\xi\|^2\end{aligned}$$

As $\mathscr{A}$ is weak*-dense, hence strongly dense in $\mathscr{M}$, the closure of $\mathscr{A}\xi$ is $\mathscr{H}$. Therefore Ω_0^V extends by continuity to an isometry $\Omega^V : \mathscr{H} \to \mathscr{H}$. Clearly, $\Omega^V\xi = \gamma_0^V(\mathbb{1})\xi^V = \xi^V$. Furthermore, for all $A \in \mathscr{A}$ we have, by (5.2):

$$\begin{aligned}\Omega^V U_t A\xi &= \Omega_0^V \alpha_t(A)\xi = \gamma_0^V \circ \alpha_t(A)\xi^V = \alpha_t^V \circ \gamma_0^V(A)\xi^V\\ &= U_t^V \gamma_0^V(A)\xi^V = U_t^V \Omega^V A\xi\end{aligned}$$

and (5.6) follows. ∎

At this point the L^1-convergence of the Dyson series comes in to prove that Ω^V is unitary.

Theorem 5.2. Let $\{\mathscr{M}, \omega, \alpha\}$ be a strongly mixing W^*-dynamical system in thermal equilibrium, and $\{\mathscr{M}, \omega^V, \alpha^V\}$ its perturbation by $V = V^* \in \mathscr{M}$. Suppose that the Dyson series involved is L^1-convergent for all elements of a weak *-dense α-invariant unital *-subalgebra $\mathscr{A}$ of $\mathscr{M}$.

Then there exists a *-automorphism γ^V of $\mathscr{M}$ with the properties

$$\omega^V \circ \gamma^V = \omega \tag{5.8}$$

and

$$\gamma^V \circ \alpha_t = \alpha_t^V \circ \gamma^V \tag{5.9}$$

Proof. From the L^1-convergence (4.1) of (3.1) it follows that for all $A \in \mathscr{A}$ the limit (5.4) exists in the norm topology. By Corollary 4.2 we have

$$\omega \circ \tilde{\gamma}_0^V(A) = \lim_{t \to \infty} \omega \circ \alpha_t^V(A) = \omega^V(A)$$

Therefore the map

$$\tilde{\Omega}_0^V : \mathscr{A}\xi^V \to \mathscr{M}\xi : A\xi^V \mapsto \tilde{\gamma}_0^V(A)\xi$$

extends to an isometry $\tilde{\Omega}^V$ on $\mathscr{H}$ by the same reasoning which showed the existence of Ω^V. On the other hand, (5.5) is seen to follow from (4.1) by looking at the $n = 1$ term only. So Ω^V exists, having the properties (5.6) and (5.7)

Now, by Theorem 4.1 we have for all A and B in $\mathscr{A}$,

$$\begin{aligned}\langle B^*\xi, \tilde{\Omega}^V A\xi^V\rangle &= \langle \xi, B\tilde{\gamma}_0^V(A)\xi\rangle = \omega\big(B\tilde{\gamma}_0^V(A)\big) = \lim_{t\to\infty}\omega\big(B\alpha_{-t}\circ\alpha_t^V(A)\big)\\ &= \lim_{t\to\infty}\omega\big(\alpha_t(B)\alpha_t^V(A)\big) = \lim_{t\to\infty}\omega^V\big(\alpha_t(B)\alpha_t^V(A)\big)\\ &= \lim_{t\to\infty}\omega^V\big(\alpha_{-t}^V\circ\alpha_t(B)A\big) = \omega^V\big(\gamma_0^V(B)A\big)\\ &= \langle\xi^V, \gamma_0^V(B)A\xi^V\rangle = \langle\gamma_0^V(B^*)\xi^V, A\xi^V\rangle = \langle\Omega^V B^*\xi, A\xi^V\rangle\end{aligned}$$

It follows that $\tilde{\Omega}^V = (\Omega^V)^*$. Being both isometric, Ω^V and $\tilde{\Omega}^V$ must be unitary and each other's inverse.

Now, define for all $M \in \mathscr{M}$,

$$\gamma^V(M) := \Omega^V M(\Omega^V)^{-1} \tag{5.10}$$

Then, by (5.6) and (5.7) we have (5.8) and (5.9), and clearly γ^V has the properties of a *-morphism. It remains to prove that $\gamma^V(\mathscr{M}) = \mathscr{M}$.

Note that for all $A, B \in \mathscr{A}$,

$$\Omega^V AB\xi = \gamma_0^V(AB)\xi^V = \gamma_0^V(A)\gamma_0^V(B)\xi^V = \gamma_0^V(A)\Omega^V B\xi$$

It follows that $\Omega^V A = \gamma_0^V(A)\Omega^V$, hence $\gamma^V(A) = \gamma_0^V(A)$, for all $A \in \mathscr{A}$. Similarly $(\gamma^V)^{-1}$ is shown to extend $\tilde{\gamma}_0^V$. On the other hand, γ^V is strongly continuous since it is spatial. Therefore

$$\gamma^V(\mathscr{M}) = \gamma^V(\mathscr{A}'') \subset \gamma^V(\mathscr{A})'' = \gamma_0^V(\mathscr{A})'' \subset \mathscr{M}'' = \mathscr{M}$$

By the same argument also $(\gamma^V)^{-1}(\mathscr{M}) \subset \mathscr{M}$, and the statement follows. ∎

6. PROOF OF THE L^1-CONVERGENCE

In this section it will be shown that the results obtained in the previous sections indeed apply to the system $\{\mathscr{M}_\beta, \omega_\beta, \alpha\}$, described in Section 2, when it is perturbed by the operator $V = w(Q_0^{\beta,\eta})$. A requirement for w will be formulated.

A good choice for the sub-*-algebra of $\mathscr{M}_\beta$ on which to prove the L^1-convergence, is the *-algebra $\mathscr{A}_{\beta,\eta}$, finitely generated by the operators

$$\{W(\lambda T_t q_\eta) \mid \lambda \in \mathbb{R}, t \in \mathbb{R}\}$$

The reason the proof works, is that the commutator of two such operators decays rapidly for increasing time separation:

$$\big\|\big[W(\lambda_1 T_{t_1} q_\eta), W(\lambda_2 T_{t_2} q_\eta)\big]\big\| \leqslant 2\big|\sin\tfrac{1}{2}\lambda_1\lambda_2\sigma(q_\eta, T_{t_2-t_1}q_\eta)\big|$$

where $t \mapsto |\sigma(q_\eta, T_t q_\eta)|$ decays exponentially:

Lemma 6.1. For all $\eta > 0$ there are positive constants a and b, such that for all $t \in \mathbb{R}$,

$$|\sigma(q_\eta, T_t q_\eta)| \leqslant a \cdot e^{-b|t|} \tag{6.1}$$

These constants necessarily satisfy

$$a > b \tag{6.2}$$

For example, in the underdamped case, characterized by $\eta < 2$, one may choose $a = (1 - \frac{1}{4}\eta^2)^{-1/2}$ and $b = \frac{1}{2}\eta$.

Proof. We omit indices η. Let $p = -q'$, as in (2.4). Then, for $t > 0$,

$$\left(\frac{d^2}{dt^2} + \eta\frac{d}{dt} + 1\right)\sigma(q, T_t q) = \left(\frac{d^2}{dt^2} + \eta\frac{d}{dt} + 1\right)\int_{-\infty}^{0} p(s)q(s-t)\,ds = 0$$

because $q'' - \eta q' + q = 0$ on $(-\infty, 0)$ [cf. (1.4)]. So $t \mapsto \sigma(q, T_t q)$ satisfies the damped oscillator equation on $(0, \infty)$, and since $\sigma(q, T_{-t}q) = -\sigma(q, T_t q)$, (6.1) follows. Now,

$$\int_0^\infty \sigma(q, T_t q) = -\int_0^\infty \left(\frac{d^2}{dt^2} + \eta\frac{d}{dt}\right)\sigma(q, T_t q)\,dt$$

$$= \left[\frac{d}{dt}\sigma(q, T_t q) + \eta\sigma(q, T_t q)\right]\Bigg|_{t=0} = \sigma(q, p) + \eta\sigma(q, q) = 1$$

On the other hand,

$$\int_0^\infty |\sigma(q, T_t q)|\,dt < \int_0^\infty a \cdot e^{-bt}\,dt = \frac{a}{b}$$

Therefore, $1 < a/b$, i.e., $a > b$. ■

The following definition will help formulate our requirement for the perturbation w of the potential.

Definition. Let $\mathcal{N}$ be the Banach space of all complex measures ν on $\mathbb{R}$ of finite total variation and satisfying

$$\nu(-S) = \overline{\nu(S)}$$

for all Borel sets $S \subset \mathbb{R}$. Let the norm $\|\nu\|_{\mathcal{N}}$ be the total variation of ν. The total variation measure will be denoted by ν^+. In particular, $\nu^+(\mathbb{R}) = \|\nu\|_{\mathcal{N}}$. Let $\hat{\mathcal{N}}$ be the linear space of real functions of the form

$$f(x) = \int_{-\infty}^{\infty} e^{i\lambda x}\nu(d\lambda), \qquad \nu \in \mathcal{N}$$

$\hat{\mathcal{N}}$ is a Banach space in the norm $\|f\|_{\hat{\mathcal{N}}} := \|\nu\|_{\mathcal{N}}$.

Theorem 6.2. Let $w \in \hat{\mathcal{N}}$ be such that also w' and w'' are in $\hat{\mathcal{N}}$. Suppose that w satisfies the inequality

$$2\|w\|_{\hat{\mathcal{N}}} + a\|w''\|_{\hat{\mathcal{N}}} < b \tag{6.3}$$

where $a, b > 0$ are given by Lemma 6.1.

Then the Dyson series, associated with the perturbation operator $w(Q_0^{\beta,\eta})$, is L^1-convergent for all $A \in \mathscr{A}_{\beta,\eta}$.

We shall prove this theorem in several steps.

Let $R(n)$, with $n \in \mathbb{N}$, denote the set of all ordered sequences $r = \{r_1, r_2, \ldots, r_p\}$ of integer numbers, satisfying

$$0 < r_1 < r_2 < \cdots < r_p < n$$

Here, p is simply the length of r. Let $R(n)$ also contain the empty sequence.

Lemma 6.3. For all $f_0, \ldots, f_n \in \mathscr{T}^\beta$,

$$\|[W(f_n), [\cdots [W(f_1), W(f_0)] \cdots]]\|$$
$$\leqslant 2^n \cdot \sum_{r \in R(n)} |\tfrac{1}{2}\sigma(f_0, f_{r_1})| \times |\tfrac{1}{2}\sigma(f_{r_1}, f_{r_2})| \times \cdots \times |\tfrac{1}{2}\sigma(f_{r_p}, f_n)|$$

Proof. Repeated use of the equality

$$[W(f), W(g)] = 2i\sin(\tfrac{1}{2}\sigma(f, g))W(f+g)$$

yields

$$\|[W(f_n), [\cdots [W(f_1), W(f_0)] \cdots]]\| = 2^n S_1 \cdots S_n$$

where

$$S_k = |\sin(\tfrac{1}{2}\sigma(f_k, f_{k-1} + \cdots + f_0))|$$

Now, let $\sigma_{jk} = \frac{1}{2}|\sigma(f_j, f_k)|$ for short. The numbers S_k satisfy two bounds:

$$S_k \leqslant 1 \tag{6.4}$$

and

$$S_k \leqslant \sum_{j=0}^{k-1} \sigma_{jk} \tag{6.5}$$

We claim that these bounds imply that

$$S_1 \cdots S_n \leqslant \sum_{r \in R(n)} \sigma_{0,r_1}\sigma_{r_1,r_2} \cdots \sigma_{r_p,n} \tag{6.6}$$

We proceed by induction. In the first place, (6.6) is valid for $n = 1$ by (6.5):

$S_1 \leqslant \sigma_{01}$. Now suppose that (6.6) holds for all n up to some integer m. Then, by (6.5) and (6.4), respectively,

$$\begin{aligned} S_1 \cdots S_m S_{m+1} &\leqslant S_1 \cdots S_m(\sigma_{0,m+1} + \cdots + \sigma_{m,m+1}) \\ &\leqslant \sigma_{0,m+1} + S_1\sigma_{1,m+1} + S_1S_2\sigma_{2,m+1} + \cdots \\ &\quad + S_1S_2 \cdots S_m\sigma_{m,m+1} \end{aligned}$$

Now, we apply the induction hypothesis (6.6) for $n = 1, \ldots, m$, and conclude that

$$S_1 \cdots S_m S_{m+1} \leqslant \sigma_{0,m+1} + \sum_{n=1}^{m}\left(\sum_{r \in R(n)} \sigma_{0,r_1} \cdots \sigma_{r_p,n}\right)\sigma_{n,m+1}$$

$$= \sum_{r \in R(m+1)} \sigma_{0,r_1}\sigma_{r_1,r_2} \cdots \sigma_{r_p,m+1}$$

We conclude that (6.6) also holds for $n = m + 1$, and the statement follows by induction on m. ■

Lemma 6.4. Let a and b be given by Lemma 6.1. Then for all $n, m \in \mathbb{N}$ and all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $t_1, \ldots, t_n \in \mathbb{R}$ and $\kappa_1, \ldots, \kappa_m \in \mathbb{R}$, $s_1, \ldots, s_m \in \mathbb{R}$ we have

$$\left\|\left[W(\lambda_n T_{t_n} q_\eta), \left[\cdots\left[W(\lambda_1 T_{t_1} q_\eta), W\left(\sum_{k=1}^{m} \kappa_k T_{s_k} q_\eta\right)\right]\cdots\right]\right]\right\|$$

$$\leqslant \left(\sum_{k=1}^{m} |\kappa_k| e^{-bs_k}\right) a|\lambda_n| e^{bt_n} \prod_{l=1}^{n-1} (2 + a\lambda_l^2) \tag{6.7}$$

Proof. Let us omit indices η, and denote $T_t q$ by q_t. An application of Lemma 6.3 with $f_0 = \sum_{k=1}^{m} \kappa_k q_{s_k}$ and $f_l = \lambda_l q_{t_l}$, $(l = 1, \ldots, n)$, gives the following upper bound for the left-hand side (l.h.s.) of (6.7):

$$(\text{l.h.s.}) \leqslant 2^n \sum_{r \in R(n)} \tfrac{1}{2}\left|\sigma\left(\sum_{k=1}^{m} \kappa_k q_{s_k}, \lambda_{r_1} q_{t_{r_1}}\right)\right|$$

$$\times \tfrac{1}{2}|\sigma(\lambda_{r_1} q_{t_{r_1}}, \lambda_{r_2} q_{t_{r_2}})| \times \cdots \times \tfrac{1}{2}|\sigma(\lambda_{r_p} q_{t_{r_p}}, \lambda_n q_{t_n})|$$

This can again be estimated by the use of the bounds on $\sigma(q_s, q_t)$, given by

$$|\sigma(q_s, q_t)| \leqslant a e^{-b|s-t|} \leqslant a e^{\pm b(s-t)}$$

as follows:

$$\begin{aligned}(\text{l.h.s.}) &\leqslant 2^n \sum_{r\in R(n)} \left\{ \tfrac{1}{2}a \sum_{k=1}^{m} |\kappa_k \lambda_{r_1}| \exp\left[b\left(t_{r_1} - s_k\right)\right]\right\} \\ &\quad \times \tfrac{1}{2}a|\lambda_{r_1}\lambda_{r_2}|\exp\left[b\left(t_{r_2}-t_{r_1}\right)\right] \times \cdots \times \tfrac{1}{2}a|\lambda_{r_p}\lambda_n|\exp\left[b\left(t_n - t_{r_p}\right)\right] \\ &= 2^n\left(\sum_{k=1}^{m}|\kappa_k|e^{-bs_k}\right)\times \tfrac{1}{2}a|\lambda_n|e^{bt_n}\times \sum_{r\in R(n)}\left(\tfrac{1}{2}a\lambda_{r_1}^2\right)\cdots\left(\tfrac{1}{2}a\lambda_{r_p}^2\right)\end{aligned}$$

Now, the sum over $R(n)$ is in fact a sum over all subsets of $\{1, \ldots, n-1\}$, and therefore

$$\sum_{r\in R(n)}\left(\tfrac{1}{2}a\lambda_{r_1}^2\right)\cdots\left(\tfrac{1}{2}a\lambda_{r_p}^2\right) = \prod_{l=1}^{n-1}\left(1+\tfrac{1}{2}a\lambda_l^2\right)$$

By distributing the 2^n over the factors, we obtain (6.7). ∎

Proof of Theorem 6.2. Let $\nu \in \mathcal{N}$ be such that $\int \exp(i\lambda x)\nu(d\lambda) = w(x)$, and let $V = w(Q_0)$. Then

$$\alpha_t(V) = w(Q_t) = \int_{-\infty}^{\infty} W(\lambda q_t)\nu(d\lambda)$$

It suffices to prove the L^1-convergence (4.1) of the Dyson series for all A of the form

$$A = W\left(\sum_{k=1}^{m}\kappa_k q_{s_k}\right) \tag{6.8}$$

Applying Lemma 6.4, we find that for all $t_1, \ldots, t_n \in \mathbb{R}$ the following holds:

$$\begin{aligned}\|V(A;t_1,\ldots,t_n)\| &:= \|\left[\alpha_{t_n}(V),\left[\cdots\left[\alpha_{t_1}(V),A\right]\cdots\right]\right]\| \\ &\leqslant \int_{\mathbb{R}}\nu^+(d\lambda_n)\cdots\int_{\mathbb{R}}\nu^+(d\lambda_1) \\ &\quad \times\left\|\left[W(\lambda_n q_{t_n}),\left[\cdots\left[W(\lambda_1 q_{t_1}),W\left(\sum\kappa_k q_{s_k}\right)\right]\cdots\right]\right]\right\| \\ &\leqslant\left(\sum_{k=1}^{m}|\kappa_k|e^{-bs_k}\right)\left(a\int_{\mathbb{R}}|\lambda_n|\nu^+(d\lambda_n)\right) \\ &\quad\times\prod_{l=1}^{n-1}\left(\int_{\mathbb{R}}(2+a\lambda_l^2)\nu^+(d\lambda_l)\right)e^{bt_n} \\ &= c_A a\|w'\|_{\hat{\mathcal{N}}}(2\|w\|_{\hat{\mathcal{N}}}+a\|w''\|_{\hat{\mathcal{N}}})^{n-1}e^{bt_n}\end{aligned}$$

Here, $c_A = \sum_{k=1}^{m}|\kappa_k|\exp(-bs_k)$ is a positive constant, determined by the choice of A in (6.8).

Now note that

$$\int_{0\geqslant t_1\geqslant \cdots \geqslant t_n} dt_1 \cdots dt_n\, e^{bt_n} = \int_{-\infty}^{0} dt_1\, e^{bt_1}\int_{-\infty}^{t_1} dt_2\, e^{b(t_2-t_1)} \cdots \times \int_{-\infty}^{t_{n-1}} dt_n\, e^{b(t_n-t_{n-1})} = \frac{1}{b^n}$$

Therefore

$$\sum_{n=0}^{\infty}\int_{0\geqslant t_1\geqslant \cdots \geqslant t_n} dt_1 \cdots dt_n\, \| V(A;t_1,\ldots,t_n)\|$$

$$\leqslant \|A\| + c_A ab^{-1}\|w'\|_{\mathcal{N}} \sum_{n=1}^{\infty} (2\|w\|_{\mathcal{N}} + a\|w''\|_{\mathcal{N}})^{n-1}/b^{n-1}$$

This series converges provided w satisfies the inequality (6.3). ■

Remark. If w satisfies (6.3), then v is strictly convex. Indeed, from (6.3) it follows by (6.2) that $\|w''\|_{\mathcal{N}} < 1$. But then,

$$|w''(x)| = \left|\int_{\mathbb{R}} (i\lambda)^2 e^{i\lambda x}\nu(d\lambda)\right| \leqslant \int_{\mathbb{R}} \lambda^2\nu^+ (d\lambda) = \|w''\|_{\mathcal{N}} < 1,$$

and

$$v''(x) = 1 + w''(x) > 0$$

7. THE SOLUTION OF THE ANHARMONIC LANGEVIN EQUATION

In this section we shall prove our still pending claim that perturbing the dynamics helps solving the quantum Langevin equation with perturbed potential.

Let us omit all indices β and η in this section, and denote $Q_0^{\beta,\eta}$ simply by Q, $\mathcal{M}_\beta$ by $\mathcal{M}$, etc. If τ is a *-automorphism of $\mathcal{M}$, we define $\tau(E(f))$ $(f\in\mathcal{S})$, in the obvious way: since $E(f)$ is the infinitesimal generator of the group $\{W(\lambda f)\}_{\lambda\in\mathbb{R}}$, let $\tau(E(f))$ be the infinitesimal generator of $\{\tau(W(\lambda f))\}_{\lambda\in\mathbb{R}}$.

We shall show that the family $\{X_t\}$, given by

$$X_t = \alpha_t(X) \qquad \text{with} \quad X = (\gamma^{w(Q)})^{-1}(Q) \tag{7.1}$$

is a solution of the Langevin equation (1.1) with potential $v(x) = \frac{1}{2}x^2 + w(x)$, provided that w satisfies the inequality (6.3). The main line of this proof is the following simple computation: Taking the limit $t\to\infty$ in (3.3) one obtains for all $A\in\mathcal{A}$,

$$\gamma^V(A) - A = -i\int_0^{\infty}\alpha_{-t}^V([V,\alpha_t(A)])\,dt \tag{7.2}$$

Then, putting $A = Q$ and $V = w(Q)$, and using the fact that $[w(Q), Q_t] = i\sigma(q, T_t q)w'(Q)$, one finds that

$$\gamma^{w(Q)}(Q) - Q = \int_0^\infty \sigma(q, T_t q)\alpha_{-t}^{w(Q)}(w'(Q))\,dt$$

Now, let $(\gamma^{\omega(Q)})^{-1}$ act on both sides to yield

$$Q - X = \int_0^\infty \sigma(q, T_t q)w'(X_{-t})\,dt$$

This is the form of the Langevin equation, taken as a starting point for the computations in Ref. 1.

Lemma 7.1. If w satisfies the inequality (6.3), then Q and $\gamma^{w(Q)}(Q)$ have the same domain. For each ψ in this common domain the following holds:

$$\gamma^{w(Q)}(Q)\psi - Q\psi = \int_0^\infty \sigma(q, T_t q)\alpha_{-t}^{w(Q)}(w'(Q))\psi\,dt \tag{7.3}$$

Proof. For $\psi \in \mathscr{H}$, consider the difference

$$\gamma_0^{w(Q)}\left[\frac{W(\lambda q) - \mathbb{1}}{i\lambda}\right]\psi - \left[\frac{W(\lambda q) - \mathbb{1}}{i\lambda}\right]\psi \tag{7.4}$$

By (7.2) this is equal to

$$\int_0^\infty (-i)(i\lambda)^{-1}\alpha_{-t}^{w(Q)}([w(Q), W(\lambda T_t q)])\psi\,dt \tag{7.5}$$

Now, let $w = \hat{\nu}$ with $\nu \in \mathscr{N}$. Then

$$\begin{aligned}
&[w(Q), W(\lambda T_t q)] \\
&\quad = \int_{-\infty}^{\infty} [W(\lambda' q), W(\lambda T_t q)]\nu(d\lambda') \\
&\quad = \int_{-\infty}^{\infty} \{\exp[-i\lambda\lambda'\sigma(q, T_t q)] - 1\} W(\lambda T_t q) W(\lambda' q)\nu(d\lambda')
\end{aligned} \tag{7.6}$$

The integrand in (7.5) is therefore bounded in norm by

$$\int_{-\infty}^{\infty} |\lambda'\sigma(q, T_t q)|\nu^+(d\lambda') = |\sigma(q, T_t q)| \cdot \|w'\|_{\mathscr{F}}$$

which is clearly independent of λ and integrable as a function of t. By (7.6), as λ tends to zero, the integrand in (7.5) tends to

$$\alpha_{-t}^{w(Q)}\left(\int_{-\infty}^{\infty} i\lambda'\sigma(q, T_t q) W(\lambda' q)\nu(d\lambda')\right)\psi = \sigma(q, T_t q)\alpha_{-t}^{w(Q)}(w'(Q))\psi$$

It follows by the dominated convergence theorem that the limit of (7.5) as $\lambda \to 0$, is equal to the right-hand side of (7.3). Since the latter is finite for all ψ, the limit as $\lambda \to 0$ of the first term of (7.4) exists for the same ψ as the limit of the second does. This means that $\gamma^{w(Q)}(Q)$ and Q have the same domain, and (7.3) holds for ψ in this domain. ■

Let $\mathscr{D}$ denote the linear span of the vectors $W(f)\xi$ with $f \in \mathscr{S}$.

We note that the inverse of the map $L: \mathscr{S} \to \mathscr{S}: f \mapsto f'' - \eta f' + f$ is given by

$$(L^{-1}g)(t) = \int_t^{\infty} \sigma(q, T_{s-t}q) g(s)\, ds = (2\eta)^{-1/2} \int_{-\infty}^{\infty} q(t-s) g(s)\, ds$$

Corollary 7.2. (Langevin equation). If w satisfies (6.3), then for all $\psi \in \mathscr{D}$ and all f $\in \mathscr{S}$,

$$\int_{-\infty}^{\infty} \{[f''(t) - \eta f'(t)] X_t \psi + f(t)[X_t \psi + w'(X_t)\psi]\}\, dt = (2\eta)^{1/2} E(f)\psi$$

Proof. By (5.10), $\gamma^{w(Q)}(Q)$ is of the form $\Omega Q \Omega^{-1}$, where Ω is unitary. In the proof of Lemma 7.1 above it was shown that $\mathrm{Dom}(\Omega Q \Omega^{-1}) = \mathrm{Dom}(Q)$. Hence $\Omega\, \mathrm{Dom}(Q) = \mathrm{Dom}(Q)$, and for all $\psi \in \mathrm{Dom}(Q)$ we have

$$Q\psi - X\psi = \int_0^{\infty} \sigma(q, T_s q) w'(X_{-s}) \psi\, ds$$

Since $\mathscr{D} \subset \mathrm{Dom}(Q)$ and $\mathscr{D}$ is invariant for time translation, we have for all $\psi \in \mathscr{D}$,

$$Q_t \psi - X_t \psi = \int_{-\infty}^{t} \sigma(q, T_{t-s} q) w'(X_s) \psi\, ds \tag{7.7}$$

Now, because of the inequality

$$\| Q_t W(f)\xi \|^2 \leqslant \|q\|^2 (1 + 4\|f\|^2)$$

$t \mapsto \| Q_t \psi \|$ is bounded for all $\psi \in \mathscr{D}$, and the integral in (7.7) is uniformly bounded in t and in ψ. Hence we may integrate (7.7) with a function $g \in \mathscr{S}$:

$$\int_{-\infty}^{\infty} g(t) X_t \psi\, dt + \int_{-\infty}^{\infty} g(t) \int_{-\infty}^{t} \sigma(q, T_{t-s} q) w'(X_s) \psi\, ds\, dt = \int_{-\infty}^{\infty} g(t) Q_t \psi\, dt$$

Putting $g = Lf$ we obtain

$$\int_{-\infty}^{\infty} (Lf)(t) X_t \psi\, dt + \int_{-\infty}^{\infty} f(s) w'(X_s) \psi\, ds = \int_{-\infty}^{\infty} (Lf)(t) Q_t \psi\, dt$$

Finally, since $\{ Q_t = E(T_t q)\}$ solves the harmonic Langevin equation:

$$\int_{-\infty}^{\infty} (Lf)(t) Q_t \psi\, dt = E(Lf * q)\psi = (2\eta)^{1/2} E(f)\psi$$

the result follows. ■

Corollary 7.3. (Return to equilibrium). If w satisfies (6.3), the solution (7.1) of the Langevin equation satisfies, for all unit vectors $\psi \in \mathscr{H}$,

$$\lim_{t\to\infty} \langle \psi, \exp(i\lambda X_t)\psi\rangle = \omega \circ (\gamma^{w(Q)})^{-1}(W(\lambda q)) = \omega^{w(Q)}(W(\lambda q)) \quad (7.8)$$

Proof. This is a consequence of the mixing property for $\{\mathscr{M}, \omega, \alpha\}$, and Theorem 5.2. ■

8. THE DIFFERENCE BETWEEN THE LIMIT STATE AND THE GIBBS STATE

In their paper,[1] Kac and Benguria considered not only the limit $t\to\infty$, but the limit $\eta\downarrow 0$ as well. Both limits taken, the probability distribution $\mu_{\beta,0,w}$, defined in (1.8) was to emerge.

Here, it turns out, nothing can be said, strictly speaking, about the latter limit. Indeed, for all nontrivial w there exists a positive value of η, below which the inequality (6.3) breaks down. We have to content ourselves with a proof that, for small values of η, the limit measure $\mu_{\beta,\eta,w}$, given by (7.8),

$$\hat{\mu}_{\beta,\eta,w}(\lambda) = \omega^{w(Q_0^{\beta,\eta})}(W(\lambda q_\eta))$$

is close to the probability distribution $\mu_{\beta,0,w}$ of the quantum harmonic oscillator.

Let the function $\Phi : [0,\infty)\to\mathbb{R}$ be defined by

$$\Phi(\beta) = \sum_{m=1}^{\infty} \frac{2\pi m/\beta^2}{\left[(2\pi m/\beta)^2 + 1\right]^2}$$

For large β, $\Phi(\beta)$ behaves like $1/4\pi$, for small β like $\zeta(3)\beta^2/(2\pi)^3$, where ζ is Riemann's zeta function.

Theorem 8.1. Let β and η be positive numbers and let $w \in \hat{\mathscr{N}}$ be such that w' and w'' are in $\hat{\mathscr{N}}$ as well.

Then, for all $\lambda \in \mathbb{R}$,

$$|\hat{\mu}_{\beta,\eta,w}(\lambda) - \hat{\mu}_{\beta,0,w}(\lambda)| \leqslant \eta\Phi(\beta)\frac{\mathrm{tr}[\exp(-\beta h)]}{\mathrm{tr}\{\exp[-\beta(h+w)]\}}(\lambda^2 + 2\beta\|w''\|_{\hat{\mathscr{N}}})e^{\beta\|w\|_{\hat{\mathscr{N}}}} \quad (8.1)$$

Proof. Let $F_{\beta,\eta}$ be the two-point function $t\mapsto\langle\xi_\beta, Q_0^{\beta,\eta}Q_t^{\beta,\eta}\xi_\beta\rangle$ of the damped harmonic quantum oscillator. Its analytic extension to the strip

$\Lambda_1^\beta = \{z \in \mathbb{C} \mid 0 \leqslant \operatorname{Im} z \leqslant \beta\}$ is given by

$$F_{\beta,\eta}(z) = \int_{-\infty}^{\infty} \frac{k}{1 - \exp(-\beta k)} \frac{2\eta}{(k^2 - 1)^2 + \eta^2 k^2} e^{ikz} \frac{dk}{2\pi} \tag{8.2}$$

Using the canonical commutation relations one derives that, for all $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ and all $t_1, \ldots, t_n \in \mathbb{R}$,

$$\omega_\beta\big(W(\lambda_0 q_\eta) W(\lambda_1 T_{t_1} q_\eta) \cdots W(\lambda_n T_{t_n} q_\eta)\big) = \exp\left[-\sum_{j,k=0}^{n} \lambda_j \lambda_k \theta(j-k) F_{\beta,\eta}(t_j - t_k)\right] \tag{8.3}$$

Here, we mean by t_0: 0, and by $\theta(j-k)$: 0 if $j < k$, 1 if $j > k$ and 1/2 if $j = k$. The right-hand side of (8.3) extends by (8.2) to a bounded and continuous function on Λ_n^β [cf. (3.4)], analytic on the interior of this region, and whose restriction to purely imaginary arguments $\{is_1, \ldots, is_n\}$ is given by

$$\exp\left(-\tfrac{1}{2} \sum_{j,k=0}^{n} \lambda_j \lambda_k F_{\beta,\eta}(i|s_j - s_k|)\right)$$

According to (3.5), this yields the following expression for the perturbed equilibrium distribution:

$$\hat{\mu}_{\beta,\eta,w}(\lambda) = \hat{\rho}_{\beta,\eta,w}(\lambda) / \hat{\rho}_{\beta,\eta,w}(0)$$

where

$$\hat{\rho}_{\beta,\eta,w}(\lambda_0) = \int_{0 \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant \beta} ds_1 \cdots ds_n \int_{\mathbb{R}^n} \nu(d\lambda_1) \cdots \nu(d\lambda_n) \times \exp\left(-\frac{1}{2} \sum_{j,k=0}^{n} \lambda_j \lambda_k F_{\beta,\eta}(i|s_j - s_k|)\right) \tag{8.4}$$

On the other hand, let $F_{\beta,0}$ be the two-point function of the simple harmonic oscillator, defined in terms of the latter's position operator x as

$$F_{\beta,0}(t) = \operatorname{tr}(e^{-\beta h} x e^{ith} x e^{-ith}) / \operatorname{tr}(e^{-\beta h}) = (\tfrac{1}{2} \coth \tfrac{1}{2} \beta) \cos t + \tfrac{1}{2} i \sin t$$

Its analytic extension is given by

$$F_{\beta,0}(z) = \tfrac{1}{2}\left[(1 - e^{-\beta})^{-1} e^{iz} - (1 - e^{\beta})^{-1} e^{-iz}\right]$$

Define the measure $\rho_{\beta,0,w}$ on $\mathbb{R}$ by

$$\hat{\rho}_{\beta,0,w}(\lambda) = \operatorname{tr}(e^{-\beta(h+w)} e^{i\lambda x}) / \operatorname{tr}(e^{-\beta h}) \tag{8.5}$$

Then one checks, using ordinary perturbation theory, that (8.4) is valid for $\eta = 0$ as well.

The following lemma gives us a grip on the right-hand side of (8.4).

Lemma 8.2. The functions $\{s,u\} \mapsto F_{\beta,\eta}(i|s-u|)$ and $\{s,u\} \mapsto F_{\beta,0}(i|s-u|) - F_{\beta,\eta}(i|s-u|)$ are positive definite kernels on $[0,\beta] \times [0,\beta]$. Moreover, for all $s \in [0,\beta]$,

$$|F_{\beta,0}(is) - F_{\beta,\eta}(is)| \leqslant 2\eta\Phi(\beta) \tag{8.6}$$

Proof of the lemma. A computation of the Fourier coefficients of $s \mapsto F_{\beta,\eta}(is)$ on $[0,\beta]$ results in the uniformly convergent series expansion

$$F_{\beta,\eta}(is) = \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \frac{e^{2\pi ims/\beta}}{(2\pi m/\beta)^2 + \eta(2\pi|m|/\beta) + 1}$$

Since the Fourier coefficients are positive, $\{s,u\} \mapsto F_{\beta,\eta}(i|s-u|)$ is positive definite. And, because the difference

$$\frac{1}{(2\pi m/\beta)^2 + 1} - \frac{1}{(2\pi m/\beta)^2 + \eta(2\pi|m|/\beta) + 1}$$
$$= \eta \frac{2\pi|m|/\beta}{\left[(2\pi m/\beta)^2 + 1\right]\left[(2\pi m/\beta)^2 + \eta(2\pi|m|/\beta) + 1\right]} \tag{8.7}$$

is positive, the difference of corresponding kernels is positive definite as well. Finally, the right-hand side of (8.7) is bounded by

$$\eta \cdot \frac{2\pi|m|/\beta}{\left[(2\pi m/\beta)^2 + 1\right]^2}$$

and (8.6) follows. ■

Proof of Theorem 8.1 (continued). Now, let us call the argument of the exp function in (8.4): $-f(\eta)$. Then Lemma 8.2 asserts that $0 \leqslant f(\eta) \leqslant f(0)$ for all $\eta \geqslant 0$. It follows that $|\exp[-f(\eta)] - \exp[-f(0)]| \leqslant f(0) - f(\eta)$. Therefore (8.4) implies that for all $\lambda_0 \in \mathbb{R}$,

$$|\hat{\rho}_{\beta,\eta,w}(\lambda_0) - \hat{\rho}_{\beta,0,w}(\lambda_0)|$$
$$\leqslant \sum_{n=0}^{\infty} \int_{0 \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant \beta} ds_1 \cdots ds_n \int_{\mathbb{R}_n} \nu^+(d\lambda_1) \cdots \nu^+(d\lambda_n)$$
$$\times \tfrac{1}{2} \sum_{j,k=0}^{n} \lambda_j \lambda_k \left[F_{\beta,0}(i|s_j - s_k|) - F_{\beta,\eta}(i|s_j - s_k|)\right] \tag{8.8}$$

Subsequently, we interchange the sum over j and k with the λ integrals, and perform the latter. If $j \neq k$, the integral over $\lambda_j\lambda_k$ yields zero, because ν^+ is a symmetric measure. If $j = k = 0$, it yields $\lambda_0^2 \cdot \|w\|_{\mathcal{K}}^n$, and if $j = k \neq 0$, it yields $\|w''\|_{\mathcal{K}} \|w\|_{\mathcal{K}}^{n-1}$. The s-integral then becomes simple to perform, because only zero remains as an argument for the functions $F_{\beta,0}$ and $F_{\beta,\eta}$; it gives a factor $\beta^n/n!$. So the right-hand side of (8.8) is equal to

$$\tfrac{1}{2}\left[F_{\beta,0}(0)-F_{\beta,\eta}(0)\right]\sum_{n=0}^{\infty}\frac{\beta^n}{n!}\left(\lambda_0^2\|w\|_{\mathcal{N}}^n+n\|w''\|_{\mathcal{N}}\|w\|_{\mathcal{N}}^{n-1}\right)$$
$$=\tfrac{1}{2}\left[F_{\beta,0}(0)-F_{\beta,\eta}(0)\right](\lambda_0^2+\beta\|w''\|_{\mathcal{N}})e^{\beta\|w\|_{\mathcal{N}}}$$

which is bounded by $\eta\Phi(\beta)(\lambda_0^2+\beta\|w''\|_{\mathcal{N}})e^{\beta\|w\|_{\mathcal{N}}}$.

To derive from the above an upper bound for the difference of the μ's instead of the ρ's, we argue as follows: If $x, y\in\mathbb{C}$ and $x_0, y_0>0$ are such that $|x|\leqslant x_0$ and $|y|\leqslant y_0$, then

$$\left|\frac{x}{x_0}-\frac{y}{y_0}\right|=\frac{1}{x_0y_0}|xy_0-yx_0|=\frac{1}{x_0y_0}|x(y_0-x_0)-x_0(y-x)|$$
$$\leqslant\frac{1}{y_0}(|y_0-x_0|+|y-x|)$$

Applying this inequality, we obtain

$$|\hat{\mu}_{\beta,\eta,w}(\lambda)-\hat{\mu}_{\beta,0,w}(\lambda)|\leqslant\eta\Phi(\beta)\frac{\lambda^2+2\beta\|w''\|_{\mathcal{N}}}{\hat{\rho}_{\beta,0,w}(0)}e^{\beta\|w\|_{\mathcal{N}}}$$

The statement (8.1) follows by (8.5). ■

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